Math 4650
Homework 4
Solutions

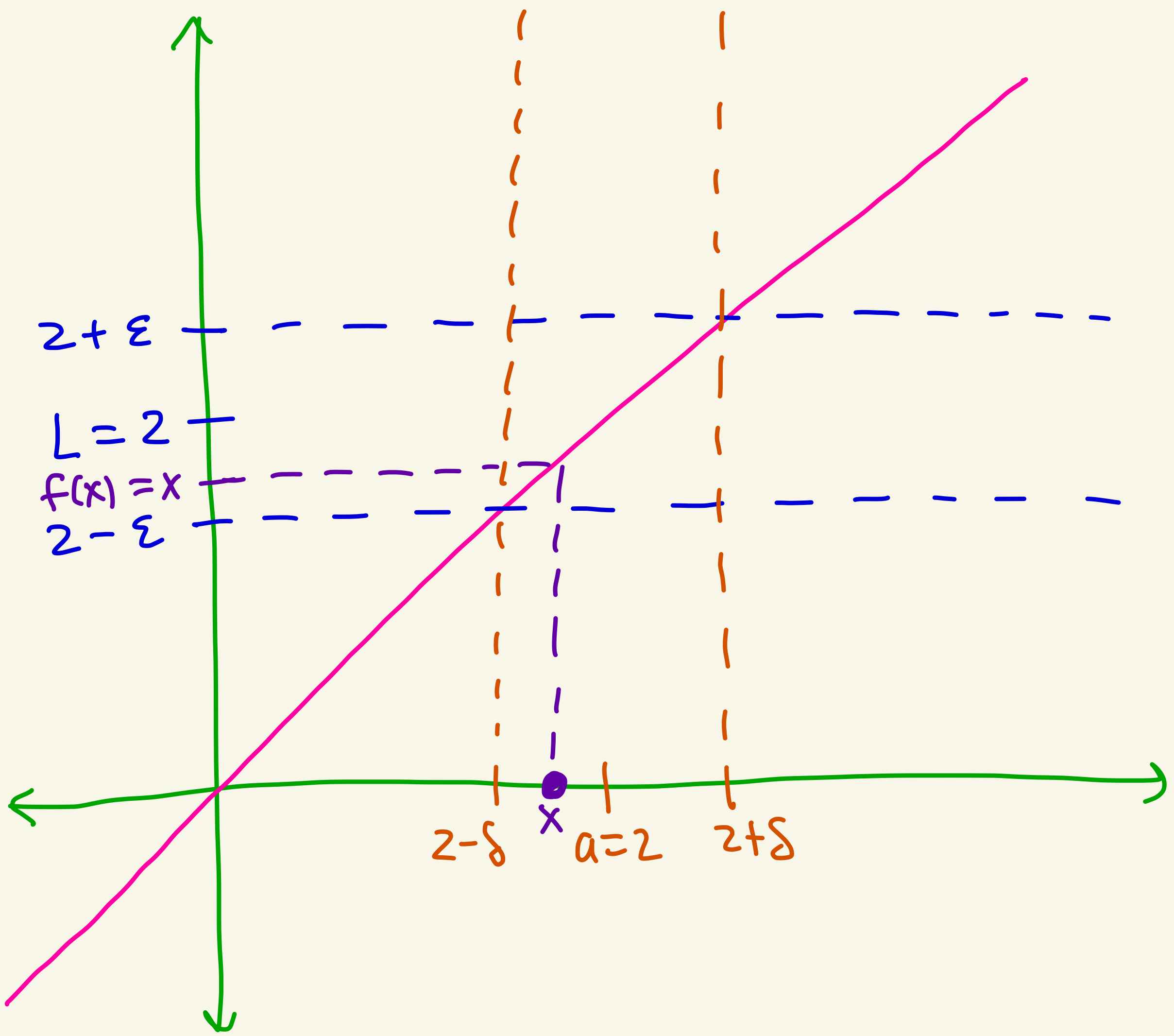
① (a)

Let $\varepsilon = 0.01$.

We want $\delta > 0$ so that if $0 < |x-2| < \delta$

then $\underbrace{|x-2| < 0.01}_{|f(x)-L| < \varepsilon}$.

Just take $\delta = 0.01$.



1(b)

Let $\varepsilon = 0.1$.
We want $\delta > 0$ where if $0 < |x-1| < \delta$
then $\left|\frac{1}{x} - 1\right| < 0.01$

$|-t| = |t|$

Note that
$$\left|\frac{1}{x} - 1\right| = \left|\frac{1-x}{x}\right| = \frac{|1-x|}{|x|} = \frac{|x-1|}{|x|}$$

If $\delta \leq \frac{1}{2}$, then $0 < |x-1| < \delta \leq \frac{1}{2}$
will give $-\frac{1}{2} < x - 1 < \frac{1}{2}$ $(x \neq 1)$
or $\frac{1}{2} < x < \frac{3}{2}$ $(x \neq 1)$.

Then, $\frac{2}{3} < \frac{1}{x} < 2$.
So, if $\delta \leq \frac{1}{2}$ then $\left|\frac{1}{x} - 1\right| = \frac{|x-1|}{|x|} < 2|x-1| < 2\delta$

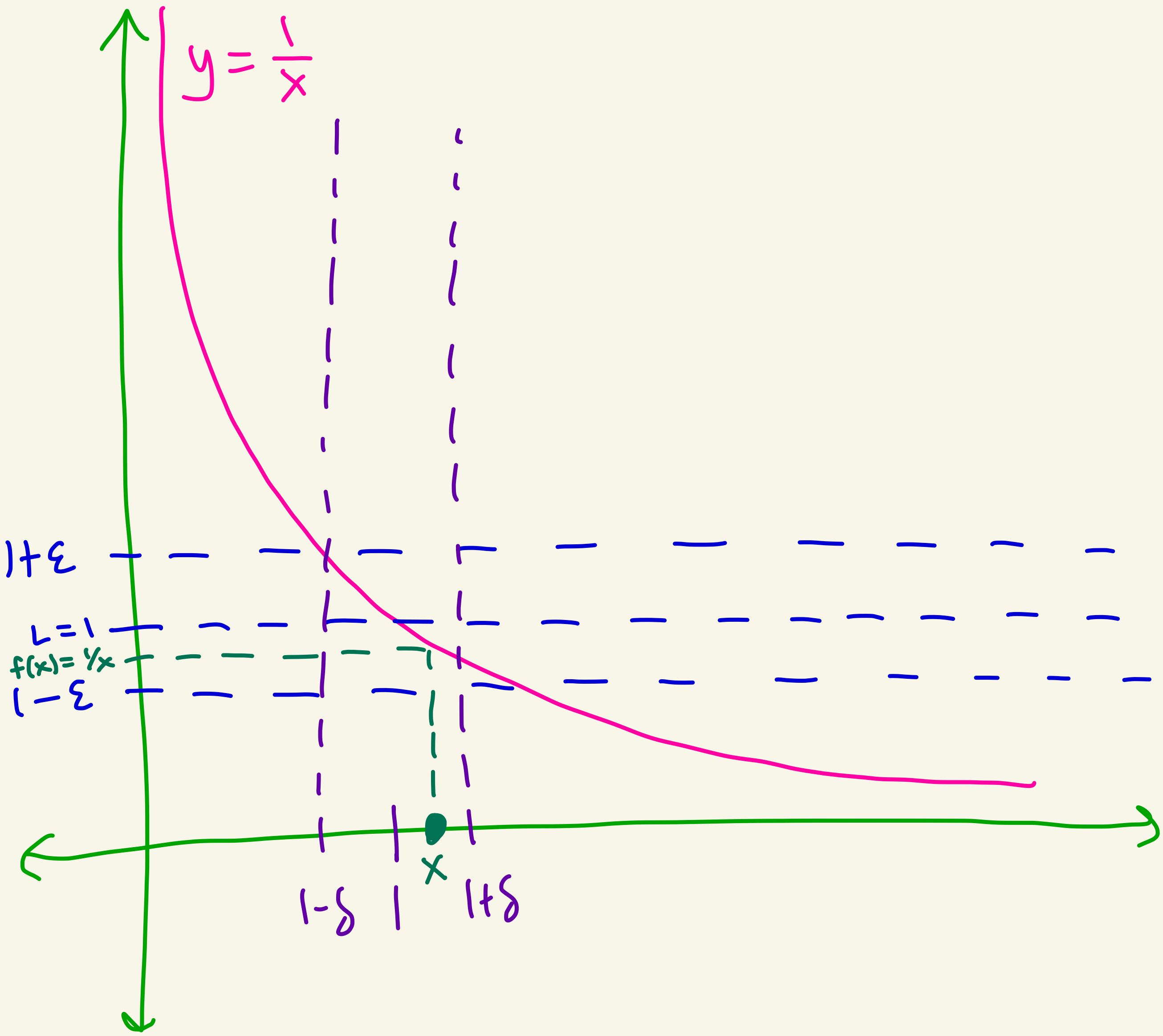
$\frac{1}{x} < 2$
$\frac{1}{|x|} < 2$

We need $2\delta \leq \underbrace{0.01}_{\varepsilon}$

Thus, pick $\delta \leq \frac{0.01}{2} = 0.005$

this is less than $1/2$ so we can just use this δ

So, if $\delta = 0.005$
and $0 < |x-1| < \underbrace{0.005}_{\delta}$
then $\left|\frac{1}{x} - 1\right| < 2\delta \leq \underbrace{0.01}_{\varepsilon}$



y = 1/x
1+ε
L = 1
f(x) = 1/x
1 − ε
1−δ
1
x
1+δ

$\boxed{2(a)}$

Let $\varepsilon > 0$.

We want to find $\delta > 0$ where if

$$0 < |x - (-1)| < \delta \quad \text{then} \quad |(2x+5) - 3| < \varepsilon.$$

That is, find $\delta > 0$ where if $0 < |x+1| < \delta$

$$\text{then} \quad |2x+2| < \varepsilon$$

Note that

$$|2x+2| = |2(x+1)| = |2| \cdot |x+1| = 2|x+1|$$

Thus, if we set $\delta = \frac{\varepsilon}{2}$ then if

$0 < |x+1| < \delta$ we get that

$$|(2x+5) - 3| = |2x+2| = 2|x+1| < 2 \cdot \frac{\varepsilon}{2} = \varepsilon$$

Therefore, $\lim_{x \to -1} (2x+5) = 3$

$\square$

$\boxed{2(b)}$

Let $\varepsilon > 0$.

We want to find $\delta > 0$ where if $0 < |x-1| < \delta$

then $\left| \frac{5x}{x+3} - \frac{5}{4} \right| < \varepsilon$.

Note that

$$\left| \frac{5x}{x+3} - \frac{5}{4} \right| = \left| \frac{20x - 5x - 15}{4x+12} \right| = \left| \frac{15x - 15}{4x+12} \right|$$

$$= \frac{|15| \cdot |x-1|}{|4x+12|} = \frac{15 \cdot |x-1|}{|4x+12|}$$

we can control $|x-1|$ with δ

lets work on $\frac{1}{|4x+12|}$

First let's assume $\delta \leq 1$

this is an arbitrary number that I picked to get a starting bound on δ so we can bound $\frac{1}{|4x+12|}$ in the above inequality

Then if $|x-1| < \delta \leq 1$

we get $-1 < x-1 < 1$.

So, $0 < x < 2$.

Thus, $0 < 4x < 8$.

And, $12 < 4x + 12 < 20$

This gives $\frac{1}{20} < \frac{1}{4x+12} < \frac{1}{12}$

Then, $\frac{1}{|4x+12|} = \frac{1}{4x+12} < \frac{1}{12}$

$\boxed{\frac{1}{4x+12} > \frac{1}{20} > 0}$

So, if $|x-1| < \delta \leq 1$, then

$$\left|\frac{5x}{x+3} - \frac{5}{4}\right| = \frac{15 \cdot |x-1|}{|4x+12|} < \frac{15}{12}|x-1| < \frac{15}{12}\delta$$

$\frac{1}{|4x+12|}$

Set $\delta = \min\left\{1, \frac{12}{15}\varepsilon\right\}$.

Then we get BOTH $\delta \leq 1$ and $\delta \leq \frac{12}{15}\varepsilon$.

So if $0 < |x-1| < \delta$, then

$$\left|\frac{5x}{x+3} - \frac{5}{4}\right| < \frac{15}{12}\delta \leq \frac{15}{12} \cdot \frac{12}{15}\varepsilon = \varepsilon.$$

from above since $\delta \leq 1$

since $\delta \leq \frac{12}{15}\varepsilon$

So, if $0 < |x-1| < \delta$, then $\left|\frac{5x}{x+3} - \frac{5}{4}\right| < \varepsilon$.

Therefore, $\lim\limits_{x \to 1} \frac{5x}{x+3} = \frac{5}{4}$.

$\blacksquare$

$\boxed{(2)(c)}$ Note that if we plug $x=2$ into x^4 then we get $2^4=16$.
So, let's try to show that $\lim_{x\to 2} x^4 = 16$.

Let $\varepsilon > 0$.
We want to find $\delta > 0$ so that if $0 < |x-2| < \delta$ then $|x^4 - 16| < \varepsilon$.

Note that

$$|x^4-16| = |x^2-4|\cdot|x^2+4|$$
$$= \underbrace{|x-2|}_{\substack{\text{this we} \\ \text{can control} \\ \text{with } \delta}} \cdot \underbrace{|x+2|\cdot|x^2+4|}_{\substack{\text{let's put a} \\ \text{starting bound} \\ \text{on } \delta \text{ to control} \\ \text{these two}}}$$

Let's start by assuming $\delta \le 1$. ← arbitrary starting bound that I picked

Suppose $0 < |x-2| < \delta \le 1$.
Then, $|x| = |x-2+2| \le |x-2| + |2| < \delta + 2 < 1+2 = 3$.
So, $|x| < 3$.
This gives

$$|x+2| \le |x| + |2| < 3+2 = 5$$

and

$$|x^2+4| \le |x^2| + |4| = |x|^2 + 4 < 3^2 + 4 = 13.$$

Thus, if $|x-2| < \delta \leq 1$, then

$$|x^4-16| = |x-2| \cdot |x+2| \cdot |x^2+4|$$
$$< |x-2| \cdot 5 \cdot 13$$
$$= 65\,|x-2|$$

Now set $\delta = \min\{1, \frac{\varepsilon}{65}\}$.

So we get BOTH $\delta \leq 1$ and $\delta \leq \frac{\varepsilon}{65}$.

Then if $0 < |x-2| < \delta$ we have

$$|x^4-16| \underset{\substack{\text{from above} \\ \text{since } \delta \leq 1}}{<} 65\,|x-2| < 65\,\delta \underset{\text{since } \delta \leq \frac{\varepsilon}{65}}{\leq} 65 \cdot \frac{\varepsilon}{65} = \varepsilon$$

Thus, if $0<|x-2|<\delta$, then $|x^4-16|<\varepsilon$

Therefore $\lim_{x \to 2} x^4 = 16$

$\blacksquare$

② (d)

Note that plugging $x=1$ into $\frac{1}{x^2}$ gives $\frac{1}{1^2}=1$.

Let's show that $\lim\limits_{x\to 1} \frac{1}{x^2}=1$.

Let $\varepsilon>0$.

We want to find $\delta>0$ so that if $0<|x-1|<\delta$ then $\left|\frac{1}{x^2}-1\right|<\varepsilon$.

$\boxed{|-t|=|t|}$

Note that

$$\left|\frac{1}{x^2}-1\right| = \left|\frac{1-x^2}{x^2}\right| = \frac{|1-x^2|}{|x^2|} = \frac{|x^2-1|}{|x^2|}$$

$$= \frac{|x-1|\cdot|x+1|}{|x^2|}$$

Let $\delta \leq \frac{1}{2}$.

Then if $|x-1|<\delta\leq\frac{1}{2}$ we get $-\frac{1}{2}<x-1<\frac{1}{2}$ or $\frac{1}{2}<x<\frac{3}{2}$.

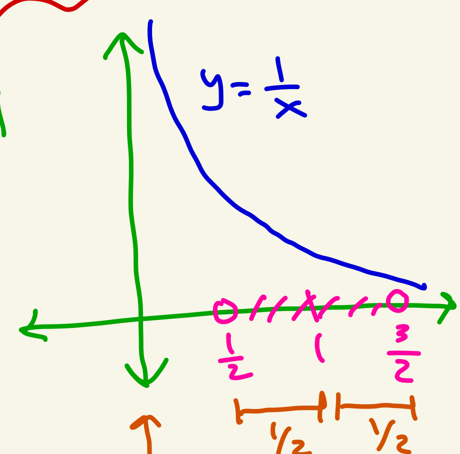
Then, $\frac{3}{2}<x+1<\frac{5}{2}$ and $\frac{1}{4}<x^2<\frac{9}{4}$.

This will give $|x+1|<\frac{5}{2}$

We can't pick $\delta<1$

We need to stay away from the asymptote at $x=0$

If you pick $\delta<1$ you'll run into issues right here



$\delta\leq 1/2$ stays away from y-axis asymptote and $\frac{1}{|x^2|} < 4$

Then, if $0 < |x-1| < \delta \leq 1$ we get

$$\left|\frac{1}{x^2} - 1\right| = |x-1| \cdot |x+1| \cdot \frac{1}{|x^2|} < |x-1| \cdot \frac{5}{2} \cdot 4$$

$$= 10 \cdot |x-1|.$$

from above

Set $\delta = \min\left\{\frac{1}{2}, \frac{\varepsilon}{10}\right\}$.

Then $\delta \leq \frac{1}{2}$ and $\delta \leq \frac{\varepsilon}{10}$.

So, if $0 < |x-1| < \delta$, then

$$\left|\frac{1}{x^2} - 1\right| < 10 \cdot |x-1| < 10\delta \leq 10 \cdot \frac{\varepsilon}{10} = \varepsilon$$

since $\delta \leq 1$ from above

since $\delta \leq \frac{\varepsilon}{10}$

Thus, if $0 < |x-1| < \delta$, then $\left|\frac{1}{x^2} - 1\right| < \varepsilon$

Therefore, $\lim\limits_{x \to 1} \frac{1}{x^2} = 1$.

$\blacksquare$

① (e)

Note that if $x=2$ then $x^3-1=7$.

Let's show that $\lim\limits_{x\to 2}(x^3-1)=7$.

Let $\varepsilon>0$.

We want to find $\delta>0$ so that if
$0<|x-2|<\delta$, then $|(x^3-1)-7|<\varepsilon$.

Note that

$$|(x^3-1)-7| = |x^3-8| = |(x-2)(x^2+2x+4)|$$
$$= |x-2|\cdot|x^2+2x+4|$$

Factor $x-2$ out of x^3-8

$$\begin{array}{r|l} & x^2+2x+4 \\ \hline (x-2) & x^3-8 \\ & -(x^3-2x^2) \\ \hline & 2x^2-8 \\ & -(2x-4x) \\ \hline & 4x-8 \\ & -(4x-8) \\ \hline & 0 \end{array}$$

this starting bound on δ is so we can bound the term $|x^2+2x+4|$

Suppose $\delta\le 1$.

Then if $|x-2|<\delta\le 1$ we get that

$$|x| = |x-2+2| \le |x-2|+|2| < \delta+2 \le 1+2 = 3$$

which gives $|x^2+2x+4| \leq |x^2|+|2x|+|4|$

triangle inequality

$$
\begin{aligned}
&= |x|^2+|2||x|+4 \\
&= |x|^2+2|x|+4 \\
&< 3^2+2\cdot 3+4 \\
&= 19
\end{aligned}
$$

Thus, if $|x-2|<\delta\leq 1$, then

$$
\begin{aligned}
|(x^3-1)-7| &= |x-2|\cdot|x^2+2x+4| \\
&< 19\cdot|x-2|
\end{aligned}
$$

Set $\delta=\min\{1,\frac{\varepsilon}{19}\}$.

Then $\delta\leq 1$ and $\delta\leq\frac{\varepsilon}{19}$.

So if $0<|x-2|<\delta$, then

$$|(x^3-1)-7| < 19\cdot|x-2| < 19\delta \leq 19\cdot\frac{\varepsilon}{19}=\varepsilon$$

since $\delta\leq 1$ from above

since $\delta\leq\frac{\varepsilon}{19}$

Thus, if $0<|x-2|<\delta$, then $|(x^3-1)-7|<\varepsilon$.

So, $\lim_{x\to 2}(x^3-1)=7$ ▨

③

Let $f: D \to \mathbb{R}$ with $a \in \mathbb{R}$ where a is a limit point of D.

Suppose $\lim_{x \to a} f(x) = L_1$ and $\lim_{x \to a} f(x) = L_2$

We will show that $L_1 = L_2$.

Let $\varepsilon > 0$.

Since $\lim_{x \to a} f(x) = L_1$ there exists $\delta_1 > 0$ where if $x \in D$ and $0 < |x - a| < \delta_1$ then $|f(x) - L_1| < \varepsilon/2$

Since $\lim_{x \to a} f(x) = L_2$ there exists $\delta_2 > 0$ where if $x \in D$ and $0 < |x - a| < \delta_2$ then $|f(x) - L_2| < \varepsilon/2$

Let $\delta = \min\{\delta_1, \delta_2\}$.

Then, $\delta \leq \delta_1$ and $\delta \leq \delta_2$.

Since a is a limit point of D there exists $\hat{x} \in D$ where $0 < |\hat{x} - a| < \delta$.

Then, $0 < |\hat{x} - a| < \delta_1$ and $0 < |\hat{x} - a| < \delta_2$.

So, $|f(\hat{x}) - L_1| < \frac{\varepsilon}{2}$ and $|f(\hat{x}) - L_2| < \frac{\varepsilon}{2}$.

Thus, $|L_1 - L_2| = |L_1 - f(\hat{x}) + f(\hat{x}) - L_2|$

$$\leq |L_1 - f(\hat{x})| + |f(\hat{x}) - L_2|$$

triangle inequality

$$= |f(\hat{x}) - L_1| + |f(\hat{x}) - L_2|$$

$|-t| = |t|$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$= \varepsilon$$

Therefore, $|L_1 - L_2| < \varepsilon$.

Since $|L_1 - L_2| \geq 0$ and $|L_1 - L_2| < \varepsilon$ for every positive ε, by HW 1, we must have that $|L_1 - L_2| = 0$.

So, $L_1 - L_2 = 0$.

Thus, $L_1 = L_2$. $\square$

4 Let $f: D \to \mathbb{R}$ where a is a limit point of D. Suppose that $\lim_{x \to a} f(x) = L$ where $L \neq 0$.

We want to find $\delta > 0$ where if $x \in D$ and $0 < |x-a| < \delta$, then $|f(x)| > 0$

Set $\varepsilon = \frac{|L|}{2} > 0$ ⟵ $\varepsilon > 0$ since $L \neq 0$

Since $\lim_{x \to a} f(x) = L$ there exists $\delta > 0$ where if $x \in D$ and $0 < |x-a| < \delta$, then $|f(x) - L| < \boxed{\frac{|L|}{2}}$ ⟵ $\boxed{\varepsilon}$

Thus, if $x \in D$ and $0 < |x-a| < \delta$, then

$$
\begin{aligned}
|L| = |L - f(x) + f(x)| &\leq |L - f(x)| + |f(x)| \\
&= |f(x) - L| + |f(x)| \\
&< \frac{|L|}{2} + |f(x)|
\end{aligned}
$$

Thus, if $x \in D$ and $0 < |x-a| < \delta$, then

$$|L| < \frac{|L|}{2} + |f(x)|$$

Thus, if $x \in D$ and $0 < |x-a| < \delta$, then $\frac{|L|}{2} < |f(x)|$

Thus, if $x \in D$ and $0 < |x-a| < \delta$, then $0 < |f(x)|$. ⟵ because $0 < \frac{|L|}{2}$

$\boxed{//}$

5

Let $f: D \to \mathbb{R}$ where a is a limit point of D and $\lim_{x\to a} f(x) = L$.

Let $\varepsilon = 1$.

Then, since $\lim_{x \to a} f(x) = L$ there exists $\delta > 0$ where if $x \in D$ and $0 < |x-a| < \delta$ then $|f(x) - L| < 1$.

So, if $x \in D$ and $0 < |x-a| < \delta$, then

$$\begin{aligned} |f(x)| &= |f(x) - L + L| \\ &\leq |f(x) - L| + |L| \\ &< 1 + |L| \end{aligned}$$

Set $M = 1 + |L|$.

Then if $x \in D$ and $0 < |x-a| < \delta$ we get $|f(x)| < M$

$\blacksquare$

⑥ (a)

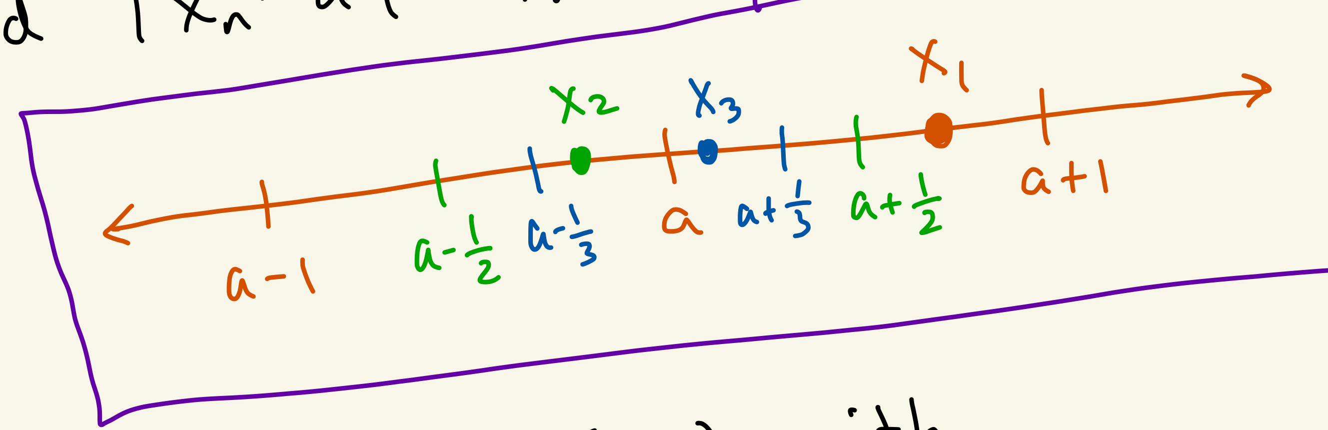
($\Rightarrow$) Suppose that a is a limit point of D. Then for every $\delta > 0$ there exists $x \in D$ with $\underbrace{0 < |x-a| < \delta}$.

means: $x \neq a$ and $|x-a| < \delta$

Set $\delta_n = \frac{1}{n}$ for $n = 1, 2, 3, 4, \ldots$

Then, for every natural number n there exists $x_n \in D$ with $x_n \neq a$ and $|x_n - a| < \frac{1}{n}$

PICTURE OF $n=1,2,3$



We have a sequence (x_n) with each $x_n \neq a$ and $x_n \in D$.

Let's show that $x_n \to a$.

Let $\varepsilon > 0$.

Pick $N > 0$ so that $\frac{1}{N} < \varepsilon$.

Then, if $n \geq N$ we get that
$$|x_n - a| < \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

Thus, $x_n \to a$.

($\Leftarrow$) Suppose that $a \in \mathbb{R}$ and there exists a sequence (x_n) where for every n we have $x_n \neq a$ and $x_n \in D$.
Further assume $x_n \to a$
Let's show this implies that a is a limit point of D.

Let $\delta > 0$.
Since $x_n \to a$ there exists an integer $N > 0$ where if $n \geq N$ then $|x_n - a| < \delta$.
In particular $|x_N - a| < \delta$.
Thus, given $\delta > 0$ we can find $x_N \in D$ with $x_N \neq a$ so that $0 < |x_N - a| < \delta$.
Thus, a is a limit point of D. $\blacksquare$

6 (b)

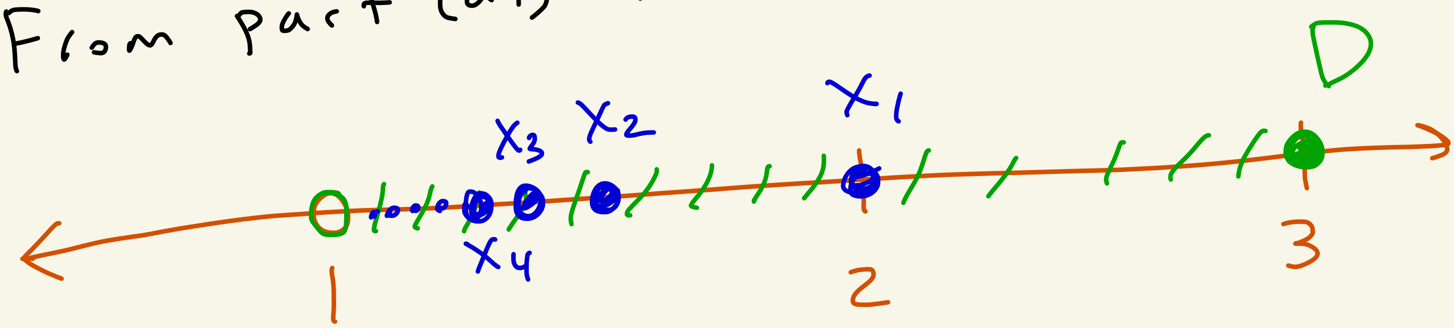
Let's show that 1 is a limit point of $D = (1, 3]$.

We use part (a).

Let $x_n = 1 + \frac{1}{n}$ for $n \geq 1$.

Then, (x_n) is a sequence of points from D with $x_n \neq 1$ for all n and $\lim_{n\to\infty} x_n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1 + 0 = 1$.

From part (a), 1 is a limit point of D.

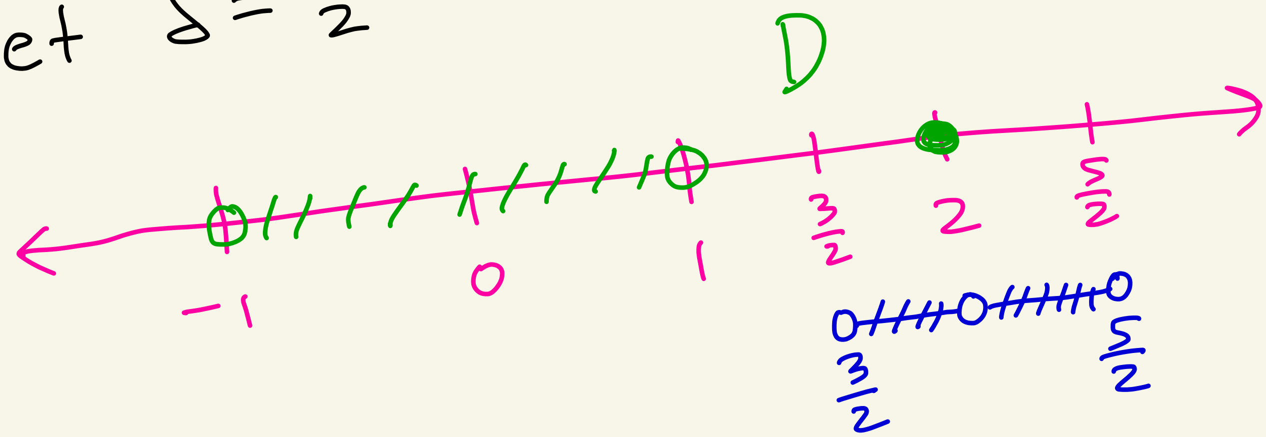


6(c)

Let $D = (-1,1) \cup \{2\}$.

We want to show that 2 is <u>not</u> a limit point of D.

Let $\delta = \frac{1}{2}$.



There does not exist $x \in D$ with $\underbrace{0 < |x-2| < \frac{1}{2}}$.

We would need:

$x \in D$ and $x \neq 2$ and $|x-2| < \frac{1}{2}$

or: $x \in D$ and $x \neq 2$ and $\frac{3}{2} < x < \frac{5}{2}$

Thus, 2 is not a limit point of D